# Lecture 14 - Orbits

## A Puzzle...

A particle of mass $m$ is subject to a force $F[t] = m\, e^{-b\,t}$. The initial position and speed are both zero. Find $x[t]$.

Solution

Newton's 2nd Law yield $F[t] = m\, e^{-b\,t} = m\, \ddot{x}[t]$, or equivalently the differential equation

$$\ddot{x}[t] = e^{-b\,t} \tag{1}$$

Now we simply need to solve this equation subject to the constraints $x[0] = 0$ and $\dot{x}[0] = 0$.

**Solution 1**: Knowing that the derivative of an exponential is an exponential, we can guess the form $x_{\text{guess}}[t] = C\, e^{-b\,t}$ whose second derivative is $\ddot{x}_{\text{guess}}[t] = C\, b^2\, e^{-b\,t}$. Thus, to match Equation (1), we set $C = \frac{1}{b^2}$ to obtain the specific solution $x_{\text{guess}}[t] = \frac{1}{b^2}\, e^{-b\,t}$.

To this specific solution, we add the general solution to $\ddot{x}_{\text{general}}[t] = 0$, which is given by $x_{\text{general}}[t] = c_1\, t + c_2$ where $c_1$ and $c_2$ are arbitrary constants. The full solution is the sum of $x_{\text{guess}}[t] + x_{\text{general}}[t]$, namely,

$$x[t] = \frac{1}{b^2}\, e^{-b\,t} + c_1\, t + c_2 \tag{2}$$

Because this solution has two arbitrary constants, we know it is the most general solution possible for this second order differential equation. At this point, we can substitute in our initial conditions. Using the initial condition $x[0] = 0$ yields $c_2 = -\frac{1}{b^2}$ while the initial condition $\dot{x}[0] = 0$ implies that $c_1 = \frac{1}{b}$. Therefore, the full solution is

$$x[t] = \frac{1}{b^2}\, e^{-b\,t} + \frac{t}{b} - \frac{1}{b^2} \tag{3}$$

**Solution 2**: We can also directly integrate the differential Equation (1). Taking the first integral,

$$\dot{x}[t] = -\frac{1}{b}\, e^{-b\,t} + c_3 \tag{4}$$

where the initial condition $\dot{x}[0] = 0$ implies that $c_3 = \frac{1}{b}$,

$$\dot{x}[t] = -\frac{1}{b}\, e^{-b\,t} + \frac{1}{b} \tag{5}$$

Taking another derivative,

$$x[t] = \frac{1}{b^2}\, e^{-b\,t} + \frac{t}{b} + c_4 \tag{6}$$

where the initial condition $x[0] = 0$ yields $c_4 = -\frac{1}{b^2}$

$$x[t] = \frac{1}{b^2}\, e^{-b\,t} + \frac{t}{b} - \frac{1}{b^2} \tag{7}$$

which agrees with the above result. □

## Gravity

### Gravitational Force

The force of gravity is given by

$$\vec{F}_{\text{grav}}[\vec{r}] = -\frac{G\,M\,m}{r^2}\,\hat{r} \tag{8}$$

where $G = 6.67 \times 10^{-11} \frac{m^3}{\text{kg}\cdot s^2}$ is the *universal gravitational constant*. Recall from Lecture 6 (Energy) that we can define a (scalar) potential for this force as

$$\begin{aligned} V_{\text{grav}}[\vec{r}] &= -\int_{\infty}^{\vec{r}} \vec{F}_{\text{grav}}[\vec{r}\,'] \cdot d\vec{r}\,' \\ &= \int_{\vec{r}}^{\vec{\infty}} \vec{F}_{\text{grav}}[\vec{r}\,'] \cdot d\vec{r}\,' \\ &= \int_{\vec{r}}^{\vec{\infty}} \left(-\frac{G\,M\,m}{(r')^2}\,\hat{r}\right) \cdot (dr'\,\hat{r}) \\ &= \int_{\vec{r}}^{\vec{\infty}} -\frac{G\,M\,m}{(r')^2}\,dr' \\ &= \left(\frac{G\,M\,m}{r'}\right)_{\vec{r}\,'=\vec{r}}^{\vec{r}\,'=\vec{\infty}} \\ &= -\frac{G\,M\,m}{r} \end{aligned} \tag{9}$$

Note that this potential only depends on the magnitude $r$ and not on the direction $\vec{r}$. We have implicitly defined the reference point of this potential to be $V_{\text{grav}}[\infty] = 0$.

For example, consider the planet Earth ($m = 6 \times 10^{24}$ kg) orbiting around the much-more-massive Sun ($M = 2 \times 10^{30}$ kg). We typically model this system by assuming that the Earth revolves around the Sun, which stays exactly fixed in space. But the Sun must also feel a gravitational pull due to the Earth. So how great of an approximation is the "fixed Sun" hypothesis?

To start off, the acceleration felt by the Earth is $m\,\vec{a}_{\text{Earth}} = -\frac{G\,M\,m}{r^2}\,\hat{r}$ or (by just considering the magnitudes of both sides)

$$a_{\text{Earth}} = \frac{G\,M}{r^2} = 6 \times 10^{-3}\,\frac{m}{s^2} \tag{10}$$

while the acceleration of the sun is

$$a_{\text{Sun}} = \frac{G\,m}{r^2} = 2 \times 10^{-8}\,\frac{m}{s^2} \tag{11}$$

Thus $\frac{a_{\text{Sun}}}{a_{\text{Earth}}} = \frac{m}{M} \ll 1$, so that it is reasonable to approximate the sun as a fixed point in space with $a_{\text{Sun}} \approx 0$. (This is the same type of assumption we make when we through a ball against a wall and assume that the wall remains stationary.)

## Near-Earth Limit: Gravitational Acceleration g

Throughout the course, we claimed that on Earth, $\vec{F}_{\text{grav}} = -m\,g\,\hat{z}$ which is a constant force pointing straight down towards the ground. Equation (8) shows that this is only an approximation. Let us find out how good of an approximation this is in our daily lives.

Example

In the limit where we are on Earth throwing a ball in the air, we expect that $r \approx R_E = 6.4 \times 10^6\,m$. Show that using $F_{\text{grav}}[\vec{z}] = -m\,g\,\hat{z}$ where $g = 9.8\,\frac{m}{s^2}$ is a good approximation for gravity in this limit.

Solution

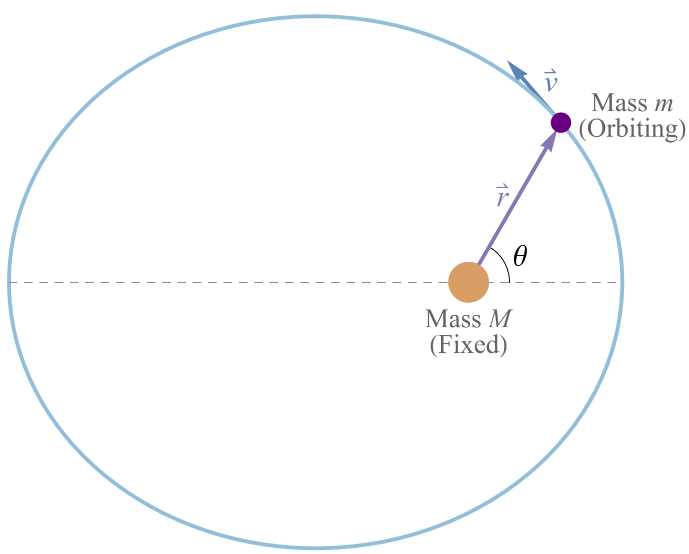
In this problem $M = 6 \times 10^{24}$ kg is the mass of the Earth and $m$ is the mass of your ball. If we through a ball to a distance $d \ll R_E = 6.4 \times 10^6\,m$ (a reasonable assumption unless you are a superhero), then the angle across the Earth spanned by this throw equals $\theta = \frac{d}{R_E} \ll 1$ and we can approximate the gravitational force as pointing straight down in the $\hat{z}$-direction. If we also assume the ball reaches a maximum height $h$ where $h \ll R_E$, then we can Taylor expand the gravitational force

$$\begin{aligned}\vec{F}_{\text{grav}}[(R_E+z)\,\hat{z}] &= m\left(-\frac{G\,M}{R_E^2}+\frac{2\,G\,M}{R_E^3}\,z+O[z]^2\right)\hat{z}\\ &= m\left(-9.8\,\frac{\text{m}}{\text{s}^2}+\left(3\times10^{-6}\,\frac{1}{\text{s}^2}\right)z+O[z]^2\right)\hat{z}\end{aligned} \tag{12}$$

Therefore, we see that to a very good approximation, gravity provides a constant acceleration of magnitude $9.8\,\frac{\text{m}}{\text{s}^2}$ straight down. Keeping the first order term would not only make equations (for example, for 2D projectile motion) much more complicated, but to see a difference you would need to keep 7 digits of precision (and our approximation $9.8\,\frac{\text{m}}{\text{s}^2}$ does not have that many digits of precision). □

## Orbits

This week, we are starting a brand new topic: gravitational orbits! For this entire lecture, we will consider a mass $m$ orbiting around a mass $M$ *fixed at the origin*.




### Central Force

Gravity has the special property that it points radially and its magnitude depends only on the distance from the source (i.e. it is spherically symmetric). Then the angular momentum of the mass $m$ has the time derivative

$$\begin{aligned}\frac{d\vec{L}}{dt} &= \frac{d}{dt}[\vec{r}\times\vec{p}]\\ &= \frac{d\vec{r}}{dt}\times\vec{p}+\vec{r}\times\frac{d\vec{p}}{dt}\\ &= \vec{r}\times\vec{F}_{\text{grav}}\\ &= \vec{0}\end{aligned} \tag{13}$$

where in the second step we have used the fact that $\frac{d\vec{r}}{dt}=\vec{v}$ is parallel to $\vec{p}=m\,\vec{v}$ and therefore the cross product is zero; in the final step we used the fact that $\vec{F}_{\text{grav}}[\vec{r}]$ and $\vec{r}$ are parallel (since $\vec{F}_{\text{grav}}$ points radially inward) and therefore their cross product is zero. Thus, *the angular momentum of a particle subject only to gravity is a constant*.

As a consequence of this fact, we will prove that a particle acting under a central force only moves in a plane. At time $t=0$, the particle has position vector $\vec{r}_0$, velocity vector $\vec{v}_0$, and an angular momentum $\vec{L}_0=m\,\vec{r}_0\times\vec{v}_0$ (which we assume to be non-zero). At any later time, the particle will be at a position $\vec{r}$ with velocity $\vec{v}$, so that the constant angular momentum vector will equal $\vec{L}_0 = m\,\vec{r}\times\vec{v}$. One of the fundamental properties of the cross product is that $\vec{r}\times\vec{v}$ is perpendicular to $\vec{r}$ and $\vec{v}$. Since $\vec{r}$ must be perpendicular to $\vec{L}_0$ at all times, $\vec{r}$ must be confined to the plane perpendicular to $\vec{L}_0$. Thus, *a particle acting only under gravity moves in a plane*.

In case you were not convinced by the previous argument, we can make it more rigorous as follows. Assume $\vec{L}_0 = \hat{z}$ points along the $z$-direction (disregarding its magnitude). Suppose for the sake of a contradiction that $\vec{r} = a\,\hat{x} + b\,\hat{y} + c\,\hat{z}$ with $c \neq 0$. If $\vec{v} = d\,\hat{x} + e\,\hat{y} + f\,\hat{z}$, then what must $d$ and $e$ be in order that $\vec{r}\times\vec{v}$ points along the $z$-direction? Using $\vec{r}\times\vec{v} = (b\,f - c\,e)\,\hat{x} + (c\,d - a\,f)\,\hat{y} + (a\,e - b\,d)\,\hat{z}$ we see that $d = \frac{a\,f}{c}$ and $e = \frac{b\,f}{c}$. But this would imply that $\vec{r}\times\vec{v} = \vec{0}$ and we assumed that the angular momentum is a non-zero vector. Therefore we must have $c = 0$.

And not to beat a dead horse, but we could also prove that a particle acting under gravity moves in a plane spanned by $\vec{r}$ and $\vec{v}$ by noting that the force acts in this plane at time 0, implying that at a time $dt$ both $\vec{r}$ and $\vec{v}$ will still be in this plane. The force will still act within this same plane, and this argument can be repeated indefinitely, proving that the particle will stay in within this plane.

## Advanced Section: Equations of Motion

## Advanced Section: Integrating the Equations of Motion

## Initial Conditions

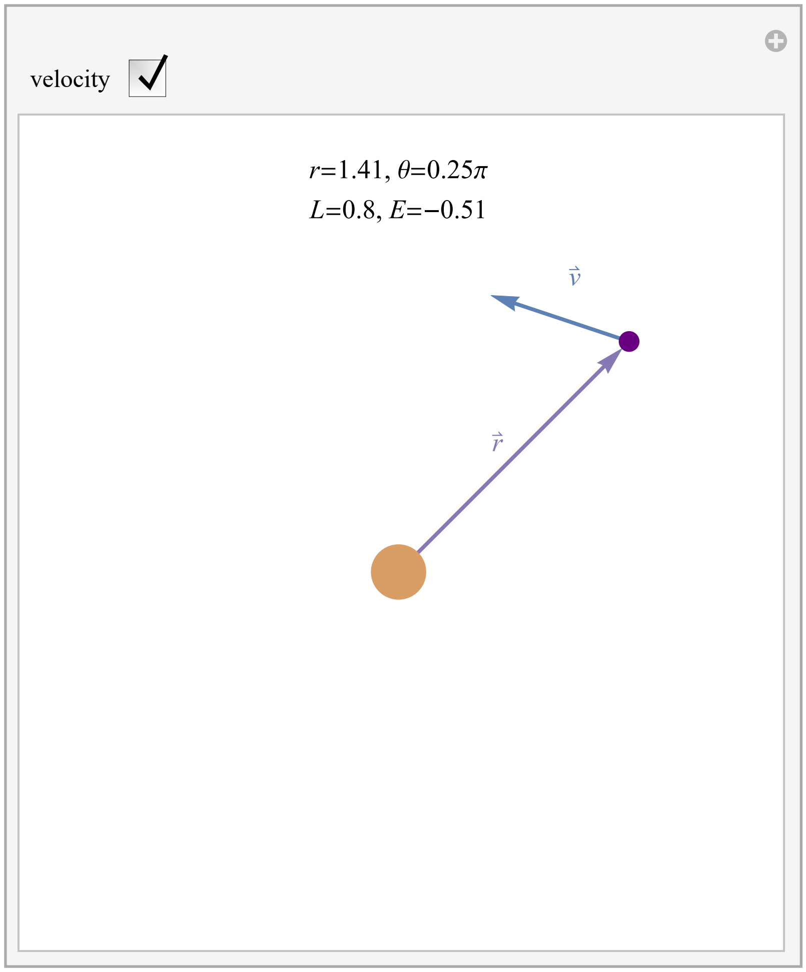
For a 1D 2nd order differential equation (such as $m\,\ddot{x} = -k\,x$), there are 2 arbitrary constants that need to be specified to determine the motion (for example, $x$ and $\dot{x}$ at time $t = 0$). For the 2D orbit problem, there are two 2nd order differential equations, and they will require 4 initial conditions to specify an orbit. For example, you could specify $r$, $\theta$, $\dot{r}$, and $\dot{\theta}$ at $t = 0$. Alternatively, you could use $r$, $\theta$, $L$, and $E$ at $t = 0$; the latter is deemed more useful because $L$ and $E$ are constants.

The following Manipulate allows you to vary $\vec{r}$ and $\vec{v}$ and see how this changes the angular momentum $L$ and the energy $E$ of the system. Recall that these are given by

$$E = \frac{1}{2}\,m\left(\dot{r}^2 + r^2\,\dot{\theta}^2\right) - \frac{G\,M\,m}{r} \tag{40}$$

$$L = m\,r^2\,\dot{\theta} \tag{41}$$

You can drag the mass $m$ (the black dot) and the vector $\vec{v}$ by clicking and dragging them with your mouse.




For example, if we rotate $v$ around in a circle (but leave its magnitude unchanged), then $E$ will be constant while $L$ will change; $L$ will reach its maximum value when $\vec{v}$ is perpendicular to $\vec{r}$ and $L = 0$ when $\vec{v}$ is parallel to $\vec{r}$.

## Advanced Section: Effective Potential

## Advanced Section: Visualizing the Effective Potential

## Orbits: Derivation

We want to solve the two equations of motion for a mass $m$ orbiting a mass $M$ fixed at the origin (assuming that $m \ll M$, this is a reasonable assumption). The equations of motion are

$$L = m\, r^2\, \dot{\theta} \tag{48}$$

$$E = \frac{1}{2}\, m\, \dot{r}^2 + \frac{L^2}{2\, m\, r^2} + V_{\text{grav}}[r] \tag{49}$$

The word "solve" is a bit ambiguous, because we could either solve for $\theta[t]$ and $r[t]$ in terms of time or for $r[\theta]$ to find the shape of the orbits instead disregarding the time dependence. We will focus on the latter.

Substituting in $V_{\text{grav}}[r] = -\frac{G\,M\,m}{r}$, we can solve the energy relation for $\dot{r}^2 = \left(\frac{dr}{dt}\right)^2$ to obtain

$$\left(\frac{dr}{dt}\right)^2 = \frac{2}{m}\left(E - \frac{L^2}{2\, m\, r^2} + \frac{G\, M\, m}{r}\right) \tag{50}$$

Since we are trying to get $r[\theta]$, we will need to cancel out the time dependence. We do so by diving this equation using the angular momentum relationship $\dot{\theta} = \frac{d\theta}{dt} = \frac{L}{m\,r^2}$ squared,

$$\frac{\left(\frac{dr}{dt}\right)^2}{\left(\frac{d\theta}{dt}\right)^2} = \left(\frac{m\,r^2}{L}\right)^2 \frac{2}{m}\left(E - \frac{L^2}{2\,m\,r^2} + \frac{G\,M\,m}{r}\right) \tag{51}$$

The left-hand side simplifies to $\left(\frac{dr}{d\theta}\right)^2$ and we can rearrange terms to obtain

$$\left(\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 = \frac{2\,m\,E}{L^2} - \frac{1}{r^2} + \frac{2\,G\,M\,m^2}{L^2\,r} \tag{52}$$

With so many $\frac{1}{r}$ variables floating around, it is natural to change variables to $y = \frac{1}{r}$ to try and simplify things. Using $dy = -\frac{1}{r^2}\,dr$,

$$\left(\frac{dy}{d\theta}\right)^2 = -y^2 + \frac{2\,G\,M\,m^2}{L^2}\,y + \frac{2\,m\,E}{L^2} \tag{53}$$

Completing the square on the right-hand side,

$$\left(\frac{dy}{d\theta}\right)^2 = -\left(y - \frac{G\,M\,m^2}{L^2}\right)^2 + \left(\frac{G\,M\,m^2}{L^2}\right)^2 + \frac{2\,m\,E}{L^2} \tag{54}$$

Changing variables again to $z = y - \frac{G\,M\,m^2}{L^2}$ and simplifying,

$$\begin{aligned}\left(\frac{dz}{d\theta}\right)^2 &= -z^2 + \left(\frac{G\,M\,m^2}{L^2}\right)^2\left(1 + \frac{2\,E\,L^2}{G^2\,M^2\,m^3}\right)\\ &\equiv -z^2 + B^2\end{aligned} \tag{55}$$

where we have defined

$$B = \frac{G\,M\,m^2}{L^2}\left(1 + \frac{2\,E\,L^2}{G^2\,M^2\,m^3}\right)^{1/2} \tag{56}$$

the solution to this differential equation is

$$z = B\,\mathrm{Cos}[\theta - \theta_0] \tag{57}$$

as can be easily verified (we could also solve for it by using separation of variables). It is customary to pick our axes so that $\theta_0 = 0$ (this just amount to rotating the final orbit), so we will drop $\theta_0$ from all further equations. Using our definitions $z = y - \frac{G\,M\,m^2}{L^2} = \frac{1}{r} - \frac{G\,M\,m^2}{L^2}$ and $B = \frac{G\,M\,m^2}{L^2}\left(1 + \frac{2\,E\,L^2}{G^2\,M^2\,m^3}\right)^{1/2}$, we obtain the orbit equation

$$\frac{1}{r} = \frac{G\,M\,m^2}{L^2}\left(1 + \epsilon\,\mathrm{Cos}[\theta]\right) \tag{58}$$

where we have defined the *eccentricity* $\epsilon$ of the orbit as

$$\epsilon \equiv \left(1 + \frac{2\,E\,L^2}{G^2\,M^2\,m^3}\right)^{1/2} \tag{59}$$

## Orbits: The Result

We have just proved that a mass $m$ orbiting a mass $M$ fixed at the origin will behave as

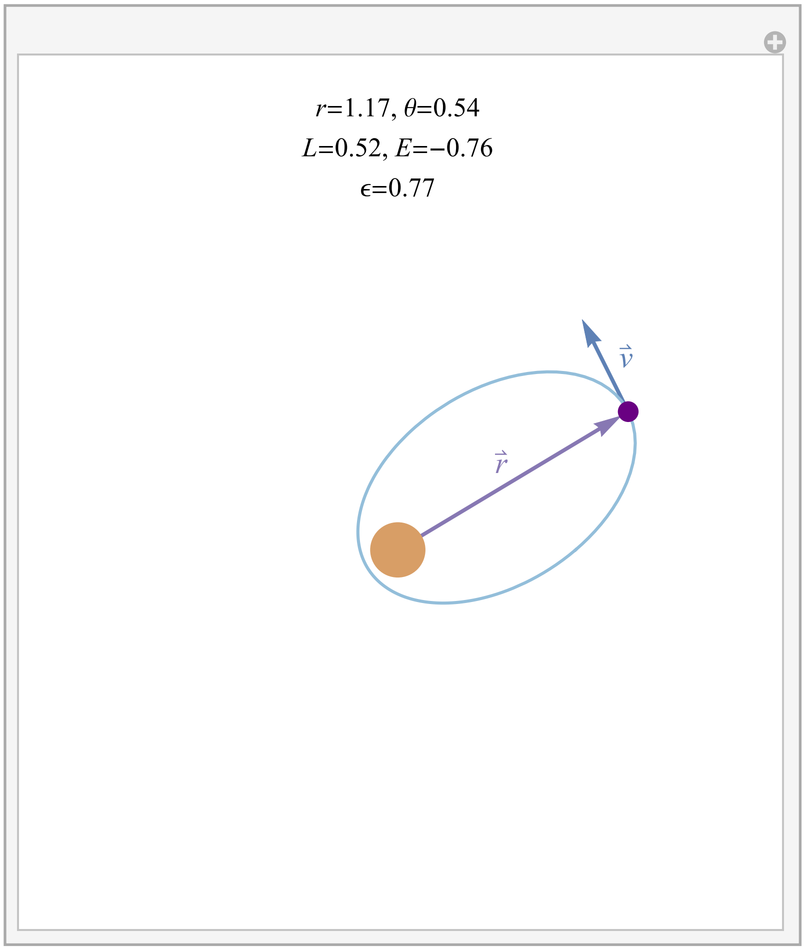
$$r[\theta] = \frac{L^2}{G\,M\,m^2}\,\frac{1}{1+\epsilon\,\mathrm{Cos}[\theta]} \tag{60}$$

where $\epsilon$ is the eccentricity

$$\epsilon \equiv \left(1 + \frac{2\,E\,L^2}{G^2\,M^2\,m^3}\right)^{1/2} \tag{61}$$

(We have chosen our axes so that the point of closest approach occurs in the $+\hat{x}$ direction.) The two variables $L$ and $E$ are fixed by initial conditions, and together with an initial location $r$ and $\theta$ at time $t = 0$, these 4 variables fully specify an orbit.

The following lecture will be dedicated solely towards understanding what these orbits look like. For now, we begin by visualizing the orbits.




Let's look at some properties of this orbit. The point of closest approach occurs at $\theta = 0$ and has a distance

$$r_{\text{min}} = \frac{L^2}{G\,M\,m^2}\,\frac{1}{1+\epsilon} \qquad (62)$$

The furthest point from the origin reached during an orbit occurs is

$$\begin{aligned} r_{\text{max}} &= \frac{L^2}{G\,M\,m^2}\,\frac{1}{1-\epsilon} \quad &(\epsilon < 1) \\ r_{\text{max}} &= \infty \quad &(\epsilon \geq 1) \end{aligned} \qquad (63)$$

where we have separated out the cases of closed orbits with $\epsilon < 1$ (circular or elliptical orbits) from open orbits with $\epsilon \geq 1$ (parabolic or hyperbolic orbits). When $\epsilon < 1$, $r_{\text{max}}$ is achieved at $\theta = \pi$.

### A Note about the Eccentricity

### Proof of Conic Orbits

## Mathematica Initialization